Determining Pareto Efficiency in Found Allocation Problems: an Approach Based on Coloured-edge Chain Graphs

Felipe Lillo[1*], Valentín Santander[2], Leidy García[2] and Lisandro Roco[3]

[1]*Deparment of Economics and Management, Universidad Católica del Maule, PO Box 617, Avenida San Miguel 3605, Talca, Chile.*

[2]*Faculty of Economics and Business, Universidad de Talca, Avenida Lircay s/n, Talca, Chile.*

[3]*Institute of Agricultural Economics, Faculty of Agricultural and Food Sciences, Universidad Austral de Chile, Valdivia 5090000, Chile.*

Abstract: The allocation of limited funds to competing activities is a well-known problem in economics and finance. Current modelling approaches for this problem are application specific and mathematically complex. This paper introduces a straightforward modelling approach based on a coloured-edge chain graph. The approach elicits a set of Pareto efficient allocations whose cardinality is theoretically studied. Additionally, the applicability of the model is illustrated through a case-study based on the chilean pension system. We conclude that despite intractability, the approach can tackle problem in practice since worst-case instances are unlikely to occur.

Keywords: Fund allocation Pareto efficient graph models coloured–edge chain graph Pareto set cardinality.

1. INTRODUCTION

Fund allocation is an important part of all business and not–for–profit organisations. Funding plans are typically established on a time basis (annual, monthly, etc.) and involve allocating anticipated income and resources among different activities or business interests. The amount of funding allocated to each area imposes restrictions on the scope of an activity's development. For example, if there is a budget reduction, some staff may have to be made redundant. Furthermore, it is worthwhile to calculate allocations that satisfy the Pareto efficient principle since economic efficiency is a desired property for social justice, Barr (1993). A related concept is the equimarginal principle. In economics, this principle posits that utility is maximised when the marginal utility of every option to its marginal price is equal to that for every other option. Therefore, an economic agent will allocate resources in such a way that this principle is satisfied, Gossen (1983).

Pareto efficiency criteria have been used for solving several optimization problems in different economic sectors and locations. Naldi et al. (2019) propose an integer linear programming model for budget allocation incorporating fairness and profit in the analyses. The main idea is to produce a fair treatment of organizational departments. Fwa and Farhan (2012) formulate and test a model to allocate budget for the maintenance of highways. They use Pareto efficiency approach to determine equitable optimal allocations. In the same way, Mahdi et al. (2019) propose a decision support system for optimal maintenance of bridges based on a family of Pareto efficient solutions. The study of Baladeh et al. (2019) determines optimal safety measures for oil and gas facilities considering budget and risk involved. Üstün and Anagün (2015) proposed a set of financial allocation strategies and multi-objective models related to building reinforcement, with the purpose of mitigating the earthquake risk of disaster in the city of Istanbul. Yadollahi et al. (2015) developed an approach to prioritize optimal Pareto solutions using a genetic algorithm to identify a unique package for bridge rehabilitation. Citanna and Siconolfi (2016) establish a theoretical model to find efficient allocations in large adverse selection economies introducing menus of contracts. Liu and Cramer (2018) compares several computational algorithms in terms of proximity to and coverage of the Pareto-optimal solutions. Kellner et al. (2019) proposed an algorithm to solve the supplier selection problem integrating risk and sustainability requirements.

One common feature throughout the literature of Pareto efficient allocations models are their strong dependence on a practical context. For example, mathematical programming approaches build models that are heavily application–specific, leaving little room for more general uses. The coloured–edge chain graph model proposed by this work is more general, and focuses on the generation of Pareto efficient allocations which can be further analyzed by post–optimal analysis techniques. As far as the literature is concerned, approaches based on graphs have been rarely used in the context of the Pareto fund allocation problem.

This work addresses the problem of fund allocation when allocation decisions are sequentially made (Mamat et al., 2014). The aim is to develop a graph model capable of determining Pareto efficient allocations. To do this, the model

*Address correspondence to this author at Deparment of Economics and Management, Universidad Católica del Maule, PO Box 617, Avenida San Miguel 3605, Talca, Chile; E-mail: flillo@ucm.cl (Felipe Lillo)

is based on a graph called the coloured–edge chain graph, a sub-group of the general coloured–edge graph introduced by Ensor and Lillo (2016). Colours are assigned to competing activities and edges model allocation alternatives. Pareto efficient allocations are calculated by applying a shortest path algorithm based on a partially ordered principle.

The remainder of manuscript is organized as follows: Section 2 introduces the coloured–edge chain graph as a model for estimating Pareto efficient allocations. Section 3 explores the use of the model in the context of fund allocation problems. Two upper bounds on the number of Pareto efficient allocations are determined in Section 3.2. The model is then tested in Section 4 to inspect the behaviour of the number of Pareto efficient allocations, and applied to a real-world case as presented in Section 5. Finally, some conclusions and further research directions are provided in Section 6.

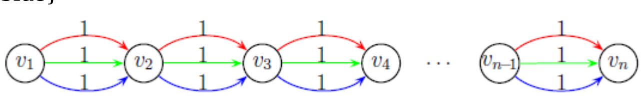
2. THE COLOURED–EDGE CHAIN GRAPH

Definition 2.1. A weighted coloured-edge chain graph G = (V, E, ω, λ) consists of a directed multi-graph (V,E) with a ordered vertex set $V = \{1,2,3,\ldots,n\}$, where n- $|v|$, an edge set E by which G only has edges $e_{uv} \in E$ for $v = u + 1$, a weight function $\omega: E \rightarrow R^+$, and a (surjective) colour function $\lambda: E \rightarrow M$, where M is a set of possible colours for the edges.

A coloured–edge chain graph uses the colour set M to represent allocation alternatives. In the context of budget allocation, such units are any kind of economic activity requiring a portion of a total available budget. Associated with each edge $e \in E$, there is an initial vertex $u \in V$ and a terminal vertex $v \in V$, a weight $\omega(e) \in R^+$, and a colour $\lambda(e) \in M$ Note that the total number of edges of a single colour is given by n-1.

The graph G is said to be finite if both V and E are finite sets, in which case M is also finite.

As a simple example, consider the following coloured–edge chain graph with three possible colours M = {red, green, blue}



Definition 2.2. Let u and v be two given vertices of G. A coloured–edge path p_{uv} is a sequence of edges of the form $\{e_{x0}x_1, e_{x1}x_2, \ldots, e_{x2x3}, \ldots, e_{xl-1x1}\}$, joining vertices $u = x_0$ and $v = x_1$, where each $x_i \in V$. The path is called simple if the vertices $x_0, x_1, \ldots, x_l$ are all distinct.

Paths in the weighted coloured–edge chain graph have an associated total weight, which is defined as follows:

Definition 2.3. For any path $p_{uv} = \{e_{x0}x_1, e_{x1}x_2, \ldots, e_{xl-1}x_l\}$ from a vertex $u = x_0$ to a vertex $v = x_i$ and any colour $c \in M$ the path weight in colour c is defined by:

$$\omega_c(p_{uv}) = \sum \lambda(e_{xi}x_{i+1}), = c\,\omega(e_{xi}x_{i+1})$$

namely the sum of the weights for those edges that have colour c.

The weight of a path is represented as a k–tuple $(\omega_{c1}(p_{uv}),\ldots, \omega_{ci}(p_{uv}),\ldots, \omega_{ck}(p_{uv}))$, giving the total weight of the path in each colour. This k–tuple is a budget allocation alternative. The total amount of resources allocated for each economic activity c_i is given by the total weight $\omega_{ci}(p_{uv})$.

From a computation standpoint, the goal is to determine paths in a coloured–edge chain graph whose weights satisfy a specific criterion. Such a criterion is established by a preference relation on path weights.

Definition 2.4. Let p_{uv} be the set of all paths from u (source) to v (destination) in G. The cardinality of this set is given by k^{n-1}. A binary relation between two paths p_{uv} and p'_{uv} in p_{uv}, is defined by $p_{uv} \leq p'_{uv}$ if and only if $\omega_c(p_{uv}) \leq \omega_c(p'_{uv})$ for all $_c$.

The relation $\leq$ is clearly reflexive and transitive and gives a partial order on the k-tuple path weights, but only a preorder on the paths themselves as multiple paths might have the same total path weight.

The imposition of a preference relation on p_{uv} produces a subset composed of only minimal paths. All tuples in this set come with a special property termed Pareto efficiency.

Definition 2.5. The set of Pareto efficient allocation paths, M_{uv}, is a set of paths joining two vertices $_u$ and $_v$ in a weighted coloured–edge chain graph such that $M_{uv} = \{ p_{uv} \in p_{uv|} \forall p'_{uv} \in p_{uv}$ with $\omega(p'_{uv}) \neq \omega(p_{uv})$, $\exists$ colour c such that $\omega_c(p_{uv}) < \omega_c(p'_{uv})\}$.

This set has an important characteristic: for any $p_{uv} \in M_{uv}$, it is impossible to determine a path p'_{uv} from u to v which has smaller weight than p_{uv} in some of its k colours without at least one of the other weights being larger, analogously to Martins (1984).

3. COLOURED–EDGE CHAIN GRAPH AND FUND ALLOCATION

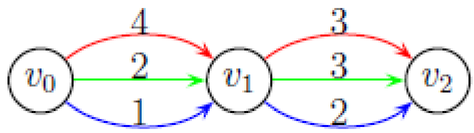
Given a fund, a time horizon and a group of activities competing for the fund, a decision maker must decide how to allocate it to activities for that period of time. According to Mamat et al. (2014), a sequential allocation process (one stage at a time) is more convenient since an investor can minimize risk by previously knowing the sequence of investment. The coloured–edge chain graph modelling approach addresses this issue, as nodes represent time stages, colours identify activities (projects, departments, areas, etc.) and coloured edges are allocation alternatives between stages.

The following case provides an application of the approach in the context of sequential fund allocation.

3.1. Sequential Investment (Kwan and Yuan, 1988)

Consider that k independent projects need to be undertaken under a time horizon T. Such a horizon is divisible in a discrete time scale so that $T = \{1,2,\ldots,i,\ldots,n\text{-}1, n\}$ (week, month, year, etc.). All projects are available at the beginning of period i so that an amount of investment must be allocated for each project between i and i+1. Note that no constraint is imposed on the amount of investment available for a period i. In terms of the coloured–edge chain graph, n+1 vertices are needed to represent time periods (period n+1 closes the process), each coloured edge between two vertices posses a real weight $\omega(e)$ that represents the investment allocated for the project. Therefore, each sequence of projects from period 1 to n+1 corresponds to a path in a coloured–edge chain graph. As a result, the path weight $(\omega_{c1}(p_{uv}), \ldots, \omega_{ci}(p_{uv}), \ldots, \omega_{ck}(p_{uv}))$ shows the total investment given to each project for time horizon T[1].

As an illustration, consider projects A, B and C to be performed within two months. For the first month, 4.0, 2.0 and 1.0 dollars are allocated for Project A (red), B (blue) and C (green), respectively. For the second month, 3.0 dollars are allocated for projects A and B, and 2.0 dollars for project C. The resulting coloured–edge chain for this case is:



Nine paths are identified from vertex v_0 to vertex v_2. The corresponding weights are (7,0,0), (4,3,0), (4,0,2), (0,5,0), (3,2,0), (0,2,2), (0,0,3), (3,0,1) and (0,3,1). After applying Definition 2.5, this set of path weights becomes (7,0,0), (0,5,0), (3,2,0), (0,2,2), (0,0,3), (3,0,1) and (0,3,1). These tuples are sequential investment options satisfying the Pareto efficient principle. Thus, tuple (3,2,0) indicates a fund of 3 dollars is given to project A, 2 dollars are for project B and no fund is assigned for project C.

3.2. Number of Pareto Efficient Allocations

This section focuses on establishing the number of Pareto efficient tuples, in other words, the cardinality of M_{uv}. The interest in such a number is related to the tractability of the approach. Most allocation problems require constraints to be made to make application–specific optimal allocations tractable. So if the set of Pareto efficient paths M_{uv} has manageable cardinality then the application–specific criteria can only be applied to this set.

The following lemma develops a bound for M_{uv} when a colored–edge chain is used to model sequential funding problems.

Lemma 3.1. Let G be a coloured–edge chain graph with n vertices and k colours for which the weights of the edge e_c from v_i to v_{i+1} satisfy the following condition:

1. For all colours c, c', the edges e_c and e_{ct} from v_i to v_{i+1} have the weight given by $\omega(e_c) = \omega(e_{ct}) = 2^{i\text{-}1}$.

Then M_{uv} has cardinality k^{n-1}.

Proof. The proof goes by induction on k. Let $h_k(n)$ be the number of minimal paths in a coloured–edge chain G with n vertices ($n > 1$)) and fixed number of colours k. The base case $h_k(1) = 1$ clearly holds. For the inductive hypothesis, assume that h_k (m) = km-1 holds for any exponential weighted coloured–edge chain with m vertices. Consider now an exponentially weighted–coloured–edge chain with m+1 vertices and consider two distinct paths $p = \{e_{12}, e_{23},..., e_{m+1\ m}\}$ and $p' = \{e_{12}', e_{23}',..., e_{m+1\ m}'\}$.

from v_1 to v_{m+1}. To show that p and p' are incomparable, consider two possible cases: (i) If $e_{m\ m+1} = e_{m\ m+1}'$ so p and p' share the same edge from v_m to v_{m+1} then $p = \{e_{12}, e_{23},..., e_{m\text{-}1\ m}\}$ and $p' = \{e_{12}', e_{23}',..., e_{m\text{-}1\ m}'\}$ are distinct paths from v_1 to v_m so by inductive hypothesis they must be incomparable. Hence p and p' are incomparable too.(ii) If $e_{m\ m+1} \neq e_{m\ m+1}'$ so the edges $e_{m\ m+1}$ and $e_{m\ m+1}'$ from v_m to v_{m+1} with weight $2^{m\text{-}1}$ have different colours c and c' then . $\omega_c(p) \geq 2^{m\text{-}1} > 1 + 2 + 4 + \ldots + 2^{m\text{-}2} \geq \omega_c(p')$ Whereas $\omega_c(p) \leq 1 + 2 + 4 + \ldots + 2^{m\text{-}2} < 2^{m\text{-}1} \leq \omega_c(p')$. Hence p and p' are incomparable and $h_k(\text{m}+1) = h_k(m) \times k = k^{(\text{m}+1)\text{-}1}$

Lemma 3.1 illustrates the intractability of the problem. However, the condition required by this lemma is very unlikely when independent random variables are used to represent fund allocations (weights).

Corollary 3.2. Suppose G is a coloured–edge chain satisfying the condition of Lemma 3.1 with k colours and n vertices and whose $k(n-1)$ edge weights are independent random variables. Then the probability that M_{uv} has cardinality $k^{(n-1)}$ is at least $((n-1)!)^{-k}$.

Proof. Note that the vertices of a path in a pure colour c can be rearranged without affecting the path weight. Then the weights $\omega_c(e_1), \omega_c(e_2), \ldots, \omega_c(e_{n\text{-}1})$ can be arranged in (n+1) ways, which are all equally likely. The desired probability is obtained when the k pure colour paths are considered.

Corollary 3.2 indicates that it is hard for an intractable case to occur in practice. This fact is advantageous for the development of algorithms capable of exploiting the structure of coloured–edge chains. However, algorithmic issues related to coloured–edge chains still need to be addressed.

A special case of the model is when weights are given by the weight function ω:E{1}which assigns the value 1 to each edge. In addition, the path weight $(\omega_{c2}(p_{uv}), \ldots, \omega_{ci}(p_{uv}), \ldots, \omega_{ck}(p_{uv}))$ represents a feasible allocation, a tuple for which . $\omega_{ci}(p_{uv}) + \ldots + \omega_{ci}(p_{uv}) + \ldots + \omega_{ck}(p_{uv}) = r$.

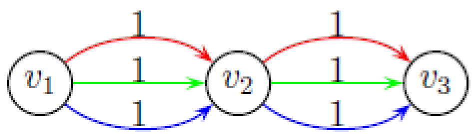
Where r is an integer positive value that could represent a number of monetary units. The set containing all feasible tuples corresponds to p_{uv}. Therefore, the Pareto efficient allocations are identified by applying definition 2.5 to p_{uv}.

Analogously, this special case can be formulated as the number of ways the integer quantity r can be partitioned so that each partition contains K elements satisfying the feasibility constraint. In this case, each partition corresponds to a feasible tuple that could (or could not) be Pareto efficient. It is noted that this resembles a subset selection and integer partition problem.

To illustrate an application, consider $r = 2$ monetary units to be allocated among $k = 3$ activities. Thus, $T = \{1,2,3\}$ corresponds to the allocation stages. The three activities are always available between two consecutive stages. The allocation plan implies that the first dollar must be allocated at Stage 1 and the second dollar allocated at Stage 2. Stage 3 closes the process. This example is modelled by a coloured–

[1] Note that by Definition 2.2 only one project is "picked" at each period. This is consistent with investment choices where "all or nothing" decisions are made.

edge chain with three colours (red, blue and green) and three vertices:



The resulting allocations are the tuples (2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1), (1,1,0), (1,0,1), (0,1,1). These tuples constitute the set P_{uv}. By applying Definition 2.5, the set of Pareto efficient allocations M_{uv} turn out to be (2,0,0), (0,2,0), (2,0,2), (1,1,0) (1,0,1), (0,1,1). Thus, the Pareto efficient tuple (2,0,0) is a fund allocation plan in which 2 monetary units are allocated to the first activity and no monetary units are allocated for the second and third activity.

To establish the cardinality of M_{uv} for Problem (2), two lemmas must first be introduced.

Lemma 3.3. Let $n \ge 0$ and $k \ge 1$. Then the following combinatorial equality holds

$$\binom{n+k-1}{k-1}+\binom{n+k-2}{k-1}+\ldots+\binom{k-1}{k-1}=\binom{k+1}{k} \qquad (3.1)$$

Proof. Use induction on n for fixed k. For $n = 0$ the equality holds since $\binom{k-1}{k-1}=1=\binom{k}{k}$. Suppose for some $m \ge 0$ that $\binom{m+k-1}{k-1}+\binom{m+k-2}{k-1}+\ldots+\binom{k-1}{k-1}=\binom{m+k}{k}$ holds.

Then for $m + 1$

$$\binom{m+1+k-1}{k-1}+\binom{m+k-1}{k-1}+\binom{m+k-2}{k-1}$$
$$+\ldots+\binom{k-1}{k-1}=\binom{m+k}{k-1}+\binom{m+k}{k}$$
$$=\frac{(m+k)!}{k!(m+1)!}(k+m+1)=\binom{m+1+k}{k}$$

Hence equality 3.1 holds for $m + 1$.

Lemma 3.4. Let $s_{nk}=\{(x_1,x_2,\ldots,x_k) \mid x_i \in Z, x_i > 0, \forall_i = 1,\ldots,k \; and \; x_i = n\}$

Then $|s_{nk}|=\binom{n+k-1}{k-1}=\frac{(n+k-1)!}{(k-1)!n!}$ (3.2)

Proof. Induction on k is used to prove that $|s_{nk}|=\binom{n+k-1}{k-1} \forall n \ge 0.$

For $k = 1$ and any n, $s_{nk}=\{(x_1) \mid x_i \in Z, x_i \ge 0 \; and \; x_1 = n\}$ so $s_{nk}=1$. For the induction hypothesis, assume for some J that $s_{nk}=\binom{n+j-1}{j-1} \forall n \ge 0$ and for any $n \ge 0$ consider the set $s_{n\,j+1}=\{(x_1,x_2,\ldots,x_{j+1}) \mid x_i \in Z, x_i \ge 0, \forall_i = 1,\ldots,k \; and \sum_{i=1}^{j+1} x_i = n\}$

For any tuple $(x_1,x_2,\ldots,x_{j+1}) \in S_{n\,j+1}$ one has that $\sum_{i=1}^{j} x_i + x_{j+1} = n$ so $x_{j+1} \le n$. Thus $(x_1,x_2,\ldots,x_j) \in S_{n-xj+1j}$, for which there are $\binom{n-x_{j+1}+j-1}{j-1}$ such tuples. So by counting over all the possible values $0,1,2,\ldots,n$ for x_{j+1}, there are $|s_{n\,j+1}|=\binom{n-0+j-1}{j-1}+\binom{n-0+j-1}{j-1}+\binom{n-2+j-1}{j-1}+\ldots+\binom{n-n+j-1}{j-1}$ distinct tuples in $S_{n\,j+1}$. Hence $S_{n\,j+1}=\binom{n+j}{j}$ by Lemma 3.3. Hence equality 3.2 is true by induction.

As a corollary of the last lemma the number of Pareto efficient allocations for a k coloured–edge chain with n vertices is obtained.

Corollary 3.5. For a coloured–edge chain graph with n vertices and k colours where edge weights equal to 1, the set of Pareto efficient allocation paths M_{uv} have cardinality $f(n, k)$ given by

$$f(n,k)=\binom{n+k-2}{k-1}. \qquad (3.3)$$

Proof. It is enough to show that any tuple $(x_1,x_2,\ldots,x_k)$, where $\sum_{i=1}^{n} xi = n-1$, is attainable by some path P in the chain and all such tuples are Pareto efficient. The path P can be constructed to have weight $(x_1,x_2,\ldots,x_k)$ by taking the first x_2 edges of the path in colour c_1, then the next x_2 edges in colour c_2 and so on. Furthermore, any two distinct tuples $(x_1,x_2,\ldots,x_k)$ and $(x_1,x_2,\ldots,x_{k,})$ must be Pareto efficient as $\sum_{i=1}^{k} x_i = n-1 = \sum_{i=1}^{k} x_i$.

Note that, if for all colours the edges in a particular colour have the same weight, then equality 3.3 still holds for a k–coloured chain with n vertices.

The pattern in $f(n, k)$ is identified by tabulating this function for small values of k and taking an arbitrary fixed n (see Table **1**).

Table 1. Values of *f* (*n*, *k*) for Several Values of *k*.

k	***f* (*n*, *k*)**
2	n
3	$\frac{n(n+1)}{2}$
4	$\frac{n(n+1)(n+2)}{6}$
5	$\frac{n(n+1)(n+2)(n+3)}{24}$

Observe that the number of Pareto efficient paths for a fixed k is given by $n^{k-1} / (k-1)$ for $k > 1$. The term n^{k-1} corresponds to a raising factorial power of n (Graham et al. 1994).

Note also that the pattern corresponds to the figurate numbers (Wunderlich, 1962). The recursive structure of these numbers can be used for the implementation of algorithms based on dynamic programming approaches. In other words, the number of Pareto efficient allocations given by $f(n, k)$ can be computed by the recurrence $f(n, k) = (n + k - 2) * f(n, k - 1)$.

3.3. Model Considerations

Using the coloured–edge chain graph as a modelling tool for fund allocation problems has the advantage of being simple and explicit. A coloured–edge chain graph is able to deliver a straightforward representation of a funding process without needing a high level of mathematical abstraction. All information can be easily displayed by just employing vertices, edges and colours.

In terms of implementation, the model works for a discrete time horizon and allocations are made in a sequential form. In addition, both the set of alternatives (colours) and time horizon (T) must be fixed. Note that a Pareto efficient path is a sequence of activities selected at each stage so that if such a path (funding plan) needs to be implemented, then only one activity must be funded at each stage. In practice, this implementation fits some fund capitalization schemes such as the Chilean pension system where individuals move their capitals from one group of financial assets to another in a regular basis. This is illustrated in a subsequent section.

Finally, the Pareto efficient tuples can be elicited by applying a shortest path algorithm that considers the chain graph G as main input. Section 4 explains more about the implementation of this algorithm.

Although the focus is the generation of Pareto efficient allocations, the resulting Pareto set can be further scrutinized by applying constraints or heuristics capable of extracting specific fund allocations. These constraints or heuristics are responsible for connecting the model to a specific application.

4. TESTING THEORETICAL RESULTS

The number of Pareto efficient allocations, M_{uv}, is investigated for coloured–edge chain graphs that meet the conditions of Problem (1) and Problem (2). The main idea is to experimentally ratify the polynomial and exponential behaviour of the number of efficient allocations.

Pareto efficient allocations are elicited by a coloured–edge chain when a shortest path algorithm is applied. However, most algorithms for shortest path problems consider the weights of the paths to be linearly ordered. In order to correctly compute Pareto efficient allocations from a coloured–edge chain graph, a shortest path algorithm must be adjusted to handle partially ordered paths.

Table 2. Exponential fit of M_{uv} Cardinality for Several Values of *k*.

k	**Exponential Function for M_{uv} Cardinality**
2	$f(n, 2) = \exp(0.6932n)$
3	$f(n, 3) = \exp(1.0986n)$
4	$f(n, 4) = \exp(1.3863n)$

For this task, the well–known Dijkstra's algorithm is adapted to process path weights that are partially ordered. Furthermore, a priority queue is used as the main data structure for storing path estimates. More information about Dijkstra's algorithm can be found in Sniedovich (2006) and Ensor and Lillo (2016), where a review of the algorithm and an adaptation for weighted coloured–edge graphs are respectively provided.

Experiments are performed for coloured–edge chains with different k–valuesand values of n between 10 and 100. The algorithm reports the cardinality of M_{uv} for each of the n vertices, but only the cardinality of the final vertex is considered in the analysis.

Fig. (**1**) shows the behavior of M_{uv} for the coloured–edge chain satisfying the condition of Lemma 3.1. A logarithm scale is used for horizontal and vertical axes. A regression analysis is performed in order to fit the curves to an exponential function[2]. Table **2** shows the functions for each value of k.

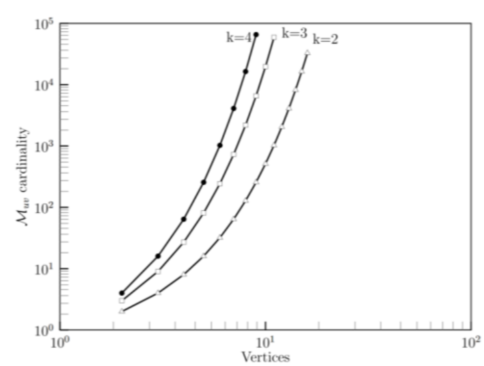


Fig. (1). Cardinality of M_{uv} for coloured–edge chain graphs with different number of colours k.

[2] A regression tool in Microsoft Excel is applied for this task.

Fig. (**2**) shows the experimental analysis for coloured–edge chains whose weights are set to 1. The order of M_{uv} cardinality is determined by applying a polynomial regression analysis between log*n* and the logarithm of the studied variable. In this way each curve is fitted to a polynomial. The exponent of the first term corresponds to the order of M_{uv} cardinality. Table **3** shows the polynomials for each value of *k*.

This numerical analysis shows that the number of competitive activities is more limiting than the number of planning periods when a sequential funding problem is computationally addressed. However, it should be noted that the condition for Lemma 3.1 will be hard to find in practical problems.

5. APPLICATION CASE: THE CHILEAN PENSION SYSTEM

The Chilean pension system is based on an individual capitalization scheme. This system relies on private companies (AFPs) that seek to improve pensions by maximizing individual capitalization funds. These companies can invest pension savings in either national or international financial assets, thereby diversifying the financial risk (Blake, 2015).

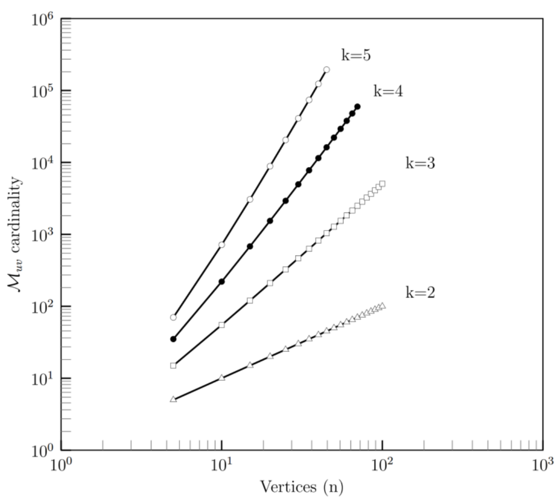


Fig. (2). Cardinality of M_{uv} for coloured-edge chain graphs with different number of colours *k*.

Table 3: Polynomial fit of M_{uv} cardinality for several values of *k*.

k	Polynomial for M_{uv} cardinality
2	$f(n, 2) = n$
3	$f(n, 3) = 0.5n^2 + 0.5n - 10^{-12}$
4	$f(n, 4) = 0.1667n^3 + 0.5n^2 + 0.3333n - 2 * 10^{-9}$
5	$f(n, 5) = 0.0417n^4 + 0.25n^3 + 0.4583n^2 + 0.25n - 2*10^{-7}$

This pension system operates by periodically allocating investments between national and international financial assets. With this in mind, the process of selecting an investment option time after time can be modelled by a coloured-edge chain graph. Once it is built, the graph can provide all investment sequences that satisfy the Pareto efficient principle without resorting to any type of mathematical programming approach. Each of these sequences is represented by a tuple that contains the total investment for each alternative. The set of all Pareto tuples (M_{uv}) can be further investigated by employing post-optimal analysis techniques. This is an important feature since Pareto efficient allocations are not necessarily optimal (Barr, 2020). Optimality implies the definition of an objective function on path weights so that tuples are transformed into a single value.

To build the coloured-edge graph model, monthly AFP investment data from 2019 is obtained[3]. The investment alternatives correspond to national (NI) as well as international (II) financial assets. Table **4** shows the investment accumulated each month for both NI and II (values are expressed in millions of dollars).

Table 4. Millions of US$ invested by AFPs in national (NI) as well as international (II) financial assets for year 2019 (Monthly data from https://www.spensiones.cl/).

Month	NI	II
Jan	114782.68	80437.47
Feb	115158.65	81733.32
Mar	115238.44	86414.53
Apr	116312.26	88684.65
May	119359.60	87135.54
Jun	122379.22	89239.36
Jul	128211.46	90224.10
Aug	131575.16	88318.24
Sep	133639.36	90316.50
Oct	127658.17	93904.53
Nov	119170.73	109071.44
Dec	128175.22	100069.20

The difference between two consecutive values are computed to determine the monthly estimate of both the investment allocated in national financial assets and the investment allocated in international financial assets (negative values are not considered). These amounts are weights in the colored edge chain graph whereas the vertices correspond to months. Once it is built, the algorithm developed by Ensor and Lillo (2016) is applied to produce the final Pareto set. Figure 3 shows the Pareto set obtained for 2019. Each tuple in the table shows the total amount invested in both national (TNI) and international (TII) financial assets. These tuples are plotted in a graph in order to describe corresponding Pareto set (see Figure **3**).

Note that a post–optimal analysis can be performed from the Pareto set provided so that the total investment is an increasing 2–ary function of the summed weight in each type of financial asset (national or international). For example, a question could be how much the investment allocated to ei-

[3] https://www.spensiones.cl/

Path	Path Weight	
	TNI	TII
P1	2982.97	4549.19
P2	13333.97	0
P3	6382.64	1153.60
P4	0	12445.63
P5	984.72	6613.39
P6	8652.75	79.78
P7	3193.55	3593.77
P8	5086.79	1529.57
P9	2280.56	6237.42
P10	7356.91	455.75
P11	4384.38	3217.8

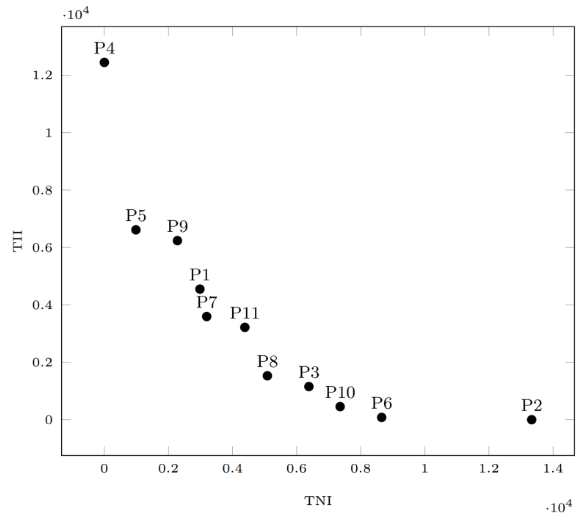


Fig. (3). Minimal path weights and corresponding Pareto set (M_{uv}) for total national (TNI) and international (TII) investments (million of US$).

ther a national or international financial asset could be increased or decreased so that the current paths remain minimal. As an illustration, if path P8 in Fig. (**3**) is picked as an investment plan (minimum total investment), an increase of over 9.0% in its relative TNI would make path P7 better. Table **5** shows path P8 obtained from the algorithm as an investment plan. An advantage of the proposed model is that different objective functions can be evaluated on paths in the Pareto set or a post–optimal analysis performed without having to rerun the algorithm.

Table 5. Path P8 as an Investment Plan.

Period	TNI	TII
Jan-Feb	0	375.97
Feb-Mar	0	79.78
Mar-Apr	0	1073.82
Apr-May	0	0
May-Jun	2103.82	0
Jun-Jul	984.71	0
Jul-Aug	0	0
Aug-Sep	1998.25	0
Sep-Oct	0	0
Oct-Nov	0	0
Nov-Dec	0	0
Total	5086.79	1529.57

6. CONCLUSIONS

In the study of fund allocation problems, a central question is whether funds may be allocated in such a way that the Pareto principle is satisfied. This paper introduced a graph modelling approach that uses a partially ordered shortest path algorithm to obtain a Pareto set of efficient allocation paths. Although a straightforward approach would be to model sequential fund allocations in which there are several investment alternatives, it does not give a new perspective and is not a truly general approach. A defining feature of our approach is the generation of a Pareto set which can be further investigated by means of a post–optimal analysis. This means a decision maker can directly apply constraints to the final Pareto set for identifying feasible allocations (tuples) without running the algorithm again.

The tractability of the approach depends on the cardinality of the final Pareto allocation set. Despite Lemma 3.1 showing an exponential order in the number of Pareto minimal paths, Corollary 3.2 supports that intractability is rare in practice unless budget allocations are set in very particular way.

A special case of the model is when graph weights are set to 1; such a case becomes useful when an integer quantity must be partitioned into Pareto efficient subsets. In practice, this quantity can be a money fund that has to be assigned to projects. This case is amenable to computation since the number of Pareto efficient paths is polinomially bounded.

Future research can tackle the application of computational implementations of the graph approach to areas outside the field of fund allocation.

CONFLICT OF INTEREST STATEMENT

The authors declare that they have no conflict of interest.

REFERENCES

Baladeh, A. E., Cheraghi, M., and Khakzad, N. (2019). A multi-objective model to optimal selection of safety measures in oil and gas facilities. Process Safety and Environmental Protection, 125:71–82.

Barr, N. (2020). Economics of the welfare state. *Oxford University Press*, USA.

Barr, N. A. (1993). The economics of the welfare state. *Stanford University Press*.

Blake, D. (2015). Issues in Pension Funding (Routledge Revivals). Routledge.

Citanna, A. and Siconolfi, P. (2016). Incentive efficient price systems in large insurance economies with adverse selection. *International Economic Review*, 57(3):1027–1056.

Ensor, A. and Lillo, F. (2016). Colored-edge graph approach for the modeling of multimodal transportation systems. *Asia-Pacific Journal of Operational Research*, 33(01):1650005.

Fwa, T. and Farhan, J. (2012). Optimal multiasset maintenance budget allocation in highway asset management. *Journal of Transportation Engineering*, 138(10):1179–1187.

Gossen, H. H. (1983). The laws of human relations and the rules of human action derived therefrom. Mit Press.

Graham, R. L., Knuth, D. E., and Patashnik, O. (1994). Concrete Mathematics: A Foundation for Computer Science. *Addison-Wesley Longman Publishing Co., Inc.*, Boston, MA, USA, 2nd edition.

Kellner, F., Lienland, B., and Utz, S. (2019). An a posteriori decision support methodology for solving the multi-criteria supplier selection problem. *European Journal of Operational Research*, 272(2):505–522.

Kwan, C. C. and Yuan, Y. (1988). Optimal sequential selection in capital budgeting: a shortcut. *Financial Management*, 1:54–59.

Liu, M. and Cramer, A. M. (2018). Computing budget allocation in multi-objective evolutionary algorithms for stochastic problems. *Swarm and Evolutionary Computation*, 38:267–274.

Mahdi, I. M., Khalil, A. H., Mahdi, H. A., and Mansour, D. M. (2019). Decision support system for optimal bridge maintenance. *International Journal of Construction Management*, pages 1–15.

Mamat, N. J. Z., Jaaman, S. H., Ahmad, R. R., and Darus, M. (2014). Fund allocation using capacitated vehicle routing problem. *In AIP Conference Proceedings*, volume 1613, pages 169–179. AIP.

Martins, E. Q. V. (1984). On a multicriteria shortest path problem. European Journal of Operational Research, 16(2):236–245.

Naldi, M., Nicosia, G., Pacifici, A., and Pferschy, U. (2019). Profit-fairness trade-off in project selection. *Socio-Economic Planning Sciences*, 67:133–146.

Sniedovich, M. (2006). Dijkstra's algorithm revisited: the dynamic programming connexion. *Control and cybernetics*, 35:599–620.

Üstün, A. K. and Anagün, A. S. (2015). Multi-objective mitigation budget allocation problem and solution approaches: The case of istanbul. *Computers & Industrial Enginee*ring, 81:118–129.

Wunderlich, M. (1962). Certain properties of pyramidal and figurate numbers. *Mathematics of Computation*, 16(80):482–486.

Yadollahi, M., Abd Majid, M. Z., and Mohamad Zin, R. (2015). Post-pareto optimality approach to enhance budget allocation process for bridge rehabilitation management. *Structure and Infrastructure Engineering*, 11(12):1565–1582.

Received: Nov 26, 2020 Revised: Dec 07, 2020 Accepted: Dec 29, 2020